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Macroscopic Stochastic Fluctuations in a One-Dimensional Mechanical System

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This paper concerns the long-time behavior of a one-dimensional mechanical system, hard rods with equal masses and lengths interacting by elastic collisions. We have noticed that for much longer times than those at which the Euler equation is valid macroscopic observables develop stochastic behavior; this might be contrasted with the expected picture based on the description of the long-time behavior of the system in terms of a Navier–Stokes correction to the Euler equation. We propose a scheme for defining an operator reminiscent of the wave operator and of the Möller morphism in scattering theory, which could be considered in one case as defining the Navier–Stokes correction, while still being meaningful when the Navier–Stokes description fails.

The model we consider is a system of hard core particles on the line moving with constant velocities except for elastic collisions. This is a system for which the hydrodynamic limit makes sense, but the Euler equation is not of the usual form, as the model has as many local conserved quantities as velocities in the system. Nevertheless, one carries out the hydrodynamic limiting procedure in the same way as for more physically realistic models: one introduces a scaling parameter ε and imposes the condition that the state at time zero varies on a spatial scale ε^{-1}. Rescaling the time by the same factor, one obtains the Euler equation in the limit $\varepsilon \to 0$.[3],[4] The degenerate nature of this model is offset by its mathematical tractability.

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What is the Navier–Stokes equation for this model? We cannot expect such a clear-cut answer as for the Euler equation, as a Navier–Stokes equation is not invariant under any space-time rescaling; therefore we cannot obtain it in a simple scaling limit. Several approaches to defining the Navier–Stokes correction are possible and are being investigated for this model and other models. Spohn[15] analyzed the covariance of the equilibrium fluctuations and showed that the leading corrections agree with the predictions of the Green–Kubo formulas. In this way he found an operator containing second derivatives in the space variable that can be identified with a Navier–Stokes correction. By adding this operator (which contains a prefactor ε) to the linearized Euler generator, he proves that the solution of the corresponding equation governs the covariance on a longer time scale.

Although this result is quite satisfactory, it only gives an indication of what the Navier–Stokes equation for this model should be. In any case, one wants to go, on one hand, beyond equilibrium results, and, on the other hand, to the full nonlinear situation. The conventional idea here would be that the Euler equation, a nonlinear hyperbolic equation, should be replaced by a parabolic equation with ε-dependent coefficients. This reflects the conventional wisdom that the state at longer times is an extremal local equilibrium state whose parameters are governed by this parabolic equation. In some models this is indeed the case.[10] On the Euler scale the Navier–Stokes correction might be determined already by the ε correction to the density profile, but this is a subtle question, as the answer might depend on the details of the initial state. On the other hand, it is not always clear what one hopes to observe in the system on a longer time scale in order to detect the effect of the Navier–Stokes correction. One case where this is clear is when there is a stationary (or traveling) solution to the Euler equation, since in this case the solution has a limit as $t \to \infty$ (in a traveling frame). If the behavior of the density profile of the particle model diverges from the predictions of the Euler equation, one may attribute this effect to a Navier–Stokes correction.

For certain models with stochastic dynamics (certain zero range and exclusion processes) the conventional picture fails.[16,1,8,9,11] The Euler equation is a "Burgers equation" which has shock wave solutions. The long-time behavior of the system is not predicted correctly by a Navier–Stokes equation. What one actually observes is that the shock profile remains rigid while developing stochastic fluctuations in time. One might conjecture that these fluctuations are a consequence of the stochastic nature of the model and so they will not develop in a mechanical system. However, we shall prove that analogous phenomena take place in the hard rod system.

It is easy to find traveling solutions of the Euler equation for the hard rod system with discrete velocities. The simplest case is for two velocities (± 1). The density profile of rods with positive velocity is

$$\rho^{+}(q)=c^{+}, \qquad 0\leqslant q\leqslant a$$

and 0 otherwise, while that for the rods with negative velocity is

$$\rho^{-}(q)=c^{-}(1-dc^{+}), \qquad 0\leqslant q\leqslant a$$

and c^{-} otherwise. Here q is the space coordinate, d is the hard rod length, and $c^{\pm}$ are positive constants less than d^{-1}, the close packing density for the hard rod system. Under the Euler evolution the profile simply shifts to the right with constant velocity

$$v_{\text{eff}}=1+\frac{2dc^{-}}{1-c^{-}d}$$

In order to see this, we need to introduce the solution operator E_t of the Euler equation as given in ref. 3. This can be written as

$$S_{-dn_t(q,v)}D_{q+vt}E_t^0C_q\rho_0=E_t\rho_0 \tag{1}$$

We explain the action of the operators appearing on the left side of Eq. (1). C_q is a nonlinear operator on positive functions (densities), which transforms a density ρ as follows. Given ρ, construct the map c_q of the line as

$$c_q(x)=x-d\sum_{v=\pm 1}\int_q^x \rho(q',v)\,dq' \tag{2}$$

Then, using c_q, transform ρ accordingly as the density of a measure. D_q is the inverse of C_q, E_t^0 is the free flow operator, S_r denotes the spatial shift by r $[(S_r\rho)(r')=\rho(r+r')]$, and $n_t(q,v)$ is a functional of the "contracted density" $C_q\rho_0$, which is the continuum analog of the number of collisions of a point particle of velocity v computed in the reduced description. Namely, it is the difference between the total masses of $C_q\rho_0$ in the regions of the phase space consisting of points that collide positively (respectively, negatively) with (q, v) during the time interval $[0, t]$. (A positive, respectively negative, collision is a collision with a particle coming from the right, respectively from the left).

We constructed the density of the traveling wave in such a way that the density of the particles with negative velocity is constant in the contracted description, so that from Eq. (1) it follows that the profile is stationary.

We now discuss the state and the time evolution for the hard rod system. Given a density profile ρ_0, we construct in a suitable way a probability measure for the hard rod system, which we denote by $i_\varepsilon \rho_0$. We first construct a Poisson state on the point particle phase space with density $C_0 \rho_0^\varepsilon$, with $\rho_0^\varepsilon(q) \equiv \rho_0(\varepsilon q)$. Then each point configuration is transformed into a hard rod configuration by acting with the analog of the dilation operator D_q, with q the location of the first point to the right of the origin. This transformation maps the Poisson measure into the measure we denoted by $i_\varepsilon \rho_0$. This is an example of a local equilibrium measure with the density profile ρ_0, the best suited for our purposes, but many other choices are possible and equally valid. The hard rod time evolution is given by a formula similar to that in Eq. (1), where ρ_0 is replaced by a hard rod configuration, q being the location of a rod with velocity v. We write T_t for the hard rod evolution operator, which does not in fact depend on the choice of q and v. For the details see ref. 3.

Previous work on this model[2] has established that in general

$$\lim_{\varepsilon \to 0} S_{\varepsilon^{-1} r} T_{\varepsilon^{-1} t} i_\varepsilon \rho_0 = \nu_{(E_t \rho_0)(r, \cdot)} \tag{3}$$

for any $r \in R$ and $t > 0$; $\nu_{f(\cdot)}$ is the extremal equilibrium measure with velocity densities f. We now consider the behavior of the state on a longer time scale.

Theorem 1. Let ρ_0 be the profile with densities $\rho^\pm(q)$ defined above. Then

$$\lim_{\varepsilon \to 0} S_{\varepsilon^{-1} r + \varepsilon^{-2} v_{\text{eff}} t} T_{\varepsilon^{-2} t} i_\varepsilon \rho_0 = \int \lambda_t(dr') \, \nu_{(E_t \rho_0)(r + r', \cdot)} \tag{4}$$

where λ_t is the law of a Brownian motion with diffusion constant equal to $v_{\text{eff}} - 1$.

Sketch of the proof. Contracting around the first rod with positive velocity and using the version of Eq. (1) for the hard rods, we see that this rod fluctuates by an amount of the order of ε^{-1} around a deterministic position: $\varepsilon^{-1} a + v_{\text{eff}} \varepsilon^{-2} t$ due to the fluctuations in the collision number. The other rods move in exactly the same way (we have neglected fluctuations of the order of $\varepsilon^{-1/2}$). By the central limit theorem the law of the fluctuations in the collision number converges to a Brownian motion with the indicated variance. The convergence of the state is a slightly more delicate matter; it follows from local central limit theorem estimates (we omit the details).

Remarks. As a corollary of Theorem 1, the density fields develop macroscopic stochastic fluctuations. This may be understood in terms of the "fluctuation theory," as discussed in Section 3 of ref. 12, where the fluctuations of the shock profile in the asymmetric simple exclusion are analyzed.

The development of stochasticity has already been seen in equilibrium fluctuations.[6] On the Euler scale the fluctuation field evolves deterministically, while, on the longer scale, it starts to move with a stochastic component. The same kind of rigidity in the motion also is observed for the fluctuation fields: spatially separated fluctuations in the density of rods with the same velocity move with the same Brownian component.

We note that if one had observed only the ensemble averages of the density field on the longer time scale, one would have missed completely the stochasticity and might have concluded that the bump in the density was simply smoothing out as time went on.

One might object that this phenomenon is an artifact of our use of discrete velocities. In fact if we replace the delta distribution concentrated on $+1$ in velocity space by a smooth density peaked at $+1$, then the bump would be dispersed and in any local observation none of these velocities would be seen. One might try to construct some kind of traveling solution periodic in time in the case of continuous velocities, but we have not investigated this point. We prefer to formulate a more general proposal. We investigate the limit as $\varepsilon \to 0$ of the following measure:

$$S_{\varepsilon^{-1}r} T_{\varepsilon^{-2}t} i_\varepsilon E_{-\varepsilon^{-2}t} \rho_0 \tag{5}$$

Notice that if the Euler evolution gives an accurate description of the particle evolution till times of the order of ε^{-2}, then the limit of (5) would define an extremal equilibrium measure with the same parameter as the "initial measure" $i_\varepsilon \rho_0$. The limiting behavior in (5) gives therefore an indication of the difference between the hard rod and the Euler evolutions in the above time scale. A comparison of two evolutions such as that proposed in (5) is commonly used in scattering theory,[14] and it has also been employed in the study of dynamical systems.[13] In principle when $\varepsilon \to 0$ we might face three possibilities: (1) the limit exists for all r, t, and ρ_0 and defines an extremal invariant measure; (2) the limit exists for all r, t, and ρ_0 and defines for some r, t, and ρ_0 a nontrivial convex combination of extremal invariant measures; (3) the limit does not exist or does not define an invariant measure, at least for some r, t, and ρ_0.

Theorem 2. In the hard rod system the limit of (5) exists for all r, t, and ρ_0 and is an invariant measure. If $\rho_0(q, v)$ is nonconstant, then for any positive t there is r for which the limit is a nontrivial convex combination of extremal invariant measures.

The proof of this theorem is completely analogous to that of Theorem 1 and is therefore omitted.

We are not at all sure whether case (2) ever occurs in more realistic models, in particular for hard cores in higher dimensions. If, however, this kind of stochasticity occurs in realistic models in situations where the Navier–Stokes correction is important, it might be relevant to interpreting some experimental results.

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